A. The idea of characteristic classes.

G

$E \xrightarrow{p} X$

We want classes

$$d(E) \in H^r(X, A).$$

$$Y \xrightarrow{f} X$$

$$f^* d(E) = d(f^* E).$$

$$\left\{ \begin{array}{l} \text{loc. triv.} \\ \text{prin. } G\text{-bundles} \end{array} \right\} \xrightarrow[\substack{\text{characteristic} \\ \text{class.}}]{} \left\{ \begin{array}{l} \text{coh. classes} \\ \text{of dim } r \\ \text{in } A \end{array} \right\}$$

Then measure how far

$E \to X$ is from $G \times X \to X$,

or how "twisted" $E \to X$ is.

Usually: $A = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

B. Characteristic classes via Yoneda.

Notation. $\boxed{H^1(X, G)} = \mathrm{Tors}_G(X)$
$$\| $$
$$[X, BG].$$

If $d \in H^r(BG, A)$, then

$E \xrightarrow{p} X$ classified by $\gamma_p : X \to BG$,

$$d(E) := \gamma_p^*(d).$$

Yoneda. $H^1(-, G) \longrightarrow H^r(-, A)$

$$\{ [-, BG] \longrightarrow [-, K(A, r)] \}.$$

$\updownarrow$ Yoneda

$$H^r(BG, A).$$

So, want to compute

$$H^*(BG).$$

Exs (a) $\mathrm{Vect}_{1,\mathbb{R}}(X) = [X, BGL_1(\mathbb{R})]$

$$\mathrm{Vect}_{1,\mathbb{R}}(X) \cong H^1(X;\mathbb{Z}/2)$$

$$[X, BGL_1(\mathbb{R})] \cong [X, BO_1]$$

$$BO_1 \cong B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2,1) \cong \mathbb{R}P^\infty$$

$$S^\infty/(\mathbb{Z}/2) \longrightarrow \mathbb{R}P^\infty$$

characteristic classes of rank 1, $\mathbb{R}$-vbs correspond to $H^*(\mathbb{R}P^\infty)$

Ex. $A = \mathbb{Z}/2$, $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[w]$, $|w| = 1$.

Fix $E \longrightarrow X$ has class $w \in H^1(X,\mathbb{Z}/2)$.

[illegible]

All other other char classes (??)

$w^n \in H^n(X, \mathbb{Z}/2)$.

<u>Ex</u>. $A = \mathbb{Z}$. $H^*(\mathbb{RP}^\infty, \mathbb{Z})$

$$\cong$$

$$\mathbb{Z}[\beta]/(2\beta), \ |\beta| = 2.$$

$$\mathbb{Z} \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \cdots$$

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$$H^i(-, \mathbb{Z}/2) \xrightarrow{\beta} H^{i+1}(-, \mathbb{Z})$$

Bockstein

$\beta(w)$ generates $H^2(\mathbb{RP}^\infty, \mathbb{Z})$.

If $E \to X$ has

$w \in H^1(X, \mathbb{Z}/2)$, the other

char classes are

$$(\beta(w))^n \in H^{2n}(X, \mathbb{Z}).$$

(b) $\mathrm{Vect}_{1,\mathbb{C}}(X) \cong [X, \mathbb{CP}^\infty] \cong H^2(X, \mathbb{Z})$

$$H^*(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}[c],$$

$$|c| = 2.$$

$$E \xrightarrow[\substack{\text{rank } 1 \\ \mathbb{C}\text{-vb}}]{} X \quad \longleftrightarrow \quad c \in H^2(X, \mathbb{Z})$$

$$c^n \in H^{2n}(X, \mathbb{Z}).$$

<u>Warning</u>: $c^n \neq c_n$.

<u>Goal</u>: understand

$$H^*(BGL_n(\mathbb{C}), \mathbb{Z}).$$

C. Main Theorem

<u>Stiefel-Whitney classes</u>. $E \to X$ be a real vector bundle. Then are classes

$H^i(X, \mathbb{Z}/2) \ni w_i(E)$, $i > 0$,

such that

(a) $f^*(w_i(E)) = w_i(f^*(E))$,

$f : Y \to X$,

Whitney sum

(b) $w(E \oplus F) = w(E) \cup w(F)$,

(Total S-W class) $w(E) = 1 + w_1(E) + w_2(E) + \cdots$

(c) $w_i(E) = 0$ if $i > \dim_{\mathbb{R}} E$,

(d) $w_1(\eta_{1,\mathbb{R}})$ is a generator of $H^1(BGL_1(\mathbb{R});\mathbb{Z}/2)$

$\cong \mathbb{Z}/2$

Universal line bundle on $BGL_1(\mathbb{R})$

Ex. A vector bundle E is stably trivial if $E \oplus \underline{\mathbb{R}}^m \cong \underline{\mathbb{R}}^n$ $(\underline{\mathbb{R}}^n = X \times \mathbb{R}^n)$

for some m, n (necessarily $\dim_{\mathbb{R}} E + m = n$).

Trivial bundles have zero S-W classes:

$w_i(\underline{\mathbb{R}}^n) = 0$ for all $i > 0$.

By (a): $\underline{\mathbb{R}}^m_X = f^* \underline{\mathbb{R}}^m_*$, $X \xrightarrow{f} *$, $H^i(*;\mathbb{Z}/2) = 0$, $i > 0$,

$w_i(\underline{\mathbb{R}}^m_X) = f^* w_i(\underline{\mathbb{R}}^m_*) = 0.$

By (b), $w(E \oplus \underline{\mathbb{R}}^m_X) = w(\underline{\mathbb{R}}^n_X) = 1 + 0 + 0 \ldots$

$\|$ $\quad\quad //$

$$\shortparallel$$

$$w(E) \cup w(\underline{\mathbb{R}}^n_X)$$

$$\shortparallel$$

$$(1+w_1(E)+w_2(E)+\dots) \cup (1+0+0\dots)$$

$$\shortparallel$$

$$1+w_1(E)+w_2(E)+\dots$$

$$\Longrightarrow \quad w_i(E)=0 \quad i>0.$$

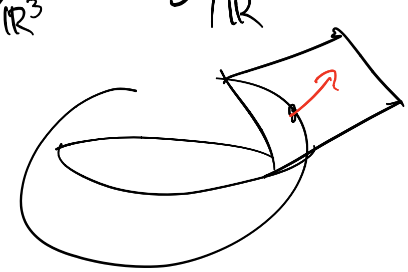
__Upshot__: stably triv. vector bundls has 0 S-W classes.

(Same is true for Chern classes.)

__Ex__. S^2

T_{S^2} rank 2 $\mathbb{R}$-v.b.

$$T_{S^2} \oplus \underline{\mathbb{R}}^1_{S^2} \cong T_{S^2/\mathbb{R}^3} \oplus N_{S^2/\mathbb{R}^3} \cong \underline{\mathbb{R}}^3_{S^2}$$



$$\Rightarrow \quad w_i(T_{S^2}) = 0.$$

But, T_{S^2} is <u>not</u> trivial

since there are no nowhere vanishing vector fields on S^2.

Ex.: M is a non-orientable differentiable manifold, then $w_1(T_M) \neq 0$ in $H^1(M, \mathbb{Z}/2)$.

$\dim M = m$

$$w_1(T_M) = w_1(\underbrace{\Lambda^m T_M}_{\mathbb{R}\text{ line bundle}})$$

$B\mathbb{Z}/2 \simeq BGL_1(\mathbb{R})$

$$\begin{array}{cc} Or_M & \Lambda^m T_M \\ \downarrow {\scriptstyle \mathbb{Z}/2} & \downarrow \\ M & M \end{array}$$

<u>Chern classes</u>

$E \longrightarrow X$ a complex v.b.

There are natural classes

$$c_i(E) \in H^{2i}(X, \mathbb{Z}), \; i \geq 0,$$

such that

(a) $f^* c_i(E) = c_i(f^* E)$,

$\qquad f : Y \to X$,

(b) $c(E \oplus F) = c(E) \cup c(F)$,

$c(E) = 1 + c_1(E) + c_2(E) + \cdots$
(total Chern class),

(c) $c_i(E) = 0$, $i > \dim_{\mathbb{C}} E$,

(d) $c_1(M_{1,\mathbb{C}})$ is a generator of

$$H^2(BGL_1(\mathbb{C}), \mathbb{Z}) \cong H^2(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}.$$

<u>Thm</u>. These exist and are unique up to choice of generator in (d).

<u>Ex</u>. T_{S^2} admits a complex structure, so it is a complex rank 1 vector bundle.

$$c_1(T_{S^2}) = \pm 2 \text{ in } H^2(S^2, \mathbb{Z}).$$

Get uniqueness by asking for

$$c_1(T_{S^2}) = 2[S^2]^\vee.$$

$\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}(-2)$

$T^1_{\mathbb{P}^1} \cong \mathcal{O}(2).$

D. Splitting off subbundles.

Thm. X be paracompact Hausdorff, E a v.b. on X, and $F \subseteq E$ a subbundle. Then, there exists $G \subseteq E$ s.t.

$$F \oplus G \xrightarrow{\cong} E.$$

In particular
$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[l, bend right] \end{tikzcd}.$$

Proof. Assume we're in the real case.

Then, $BO_n \simeq BGL_n(\mathbb{R})$.

So, since X is para. cpt. H, there exists a nondegenerate symm. bilinear form on E. $\langle -,- \rangle$

Define $G = F^{\perp}$.

Then, $F \oplus F^{\perp} \cong E$.

Ex. $0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$

obstruction to splitting is

$D^b(X)$ $\quad \mathcal{O}_X \to (E \to \mathcal{O}_X \to \mathcal{O}_X[1])$

$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$

$H^1(X, \mathcal{O}_X)$

Ex. $E/\mathbb{C}$ is an elliptic curve, $H^1(E, \mathcal{O}_E) \cong \mathbb{C}$.

$W \to GL_n(\mathbb{R})$ $\qquad X = U \cup V$

$W = U \cap V$

$W \to O_n$

$(\mathbb{R}^n_U, \langle -, - \rangle)$, $(\mathbb{R}^n_V, \langle -, - \rangle)$

$$[X, BO_n]$$
$$\cong$$
$$[X, BGL_n(\mathbb{R})]$$